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A BOUNDARY INTEGRAL EQUATION METHOD FOR CONSOLIDATION PROBLEMS

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Abstract—A time domain direct boundary integral equation method (BIEM) for Biot's linear theory of consolidation is considered. The potential representation of the solution is used not only for obtaining numerical solutions but also for investigating the behavior of the solution. The obtained information concerning the behavior of the solution is utilized to implement an accurate BIEM. A numerical example confirms the applicability of the present formulation.

1. INTRODUCTION

Analysis of consolidation nowadays is synonymous to the deformation analysis of a skeleton–fluid system. Therefore, many important problems in soil mechanics require analysis of this type since the coupling of the soil skeleton deformation and the pore fluid motion characterizes the mechanical behavior of soil. Among various theories to describe this coupling, Biot's linear theory (Biot, 1941) is popular because of its relative simplicity and generality. A number of numerical methods have been proposed for this theory so far, among which finite element methods are well accepted (Sandhu and Wilson, 1969). However, these methods are not without accuracy problems. For example, some FEM formulations are known to have difficulty in computing early time solutions. It would therefore be worthwhile to develop other numerical methods of solution for Biot's theory. This motivated us to investigate a BIEM in Biot's theory.

BIEM refers to a method of numerical analysis based on a certain potential representation of the solution of a certain mathematical problem. As the name implies, it converts the original problem into an integral equation defined on the boundary of the domain under consideration. So far the reduced dimensionality thus achieved has been emphasized as an advantage of this approach. In the present authors' opinion, however, equally important is the fact that the potential representations used in BIEM are essentially exact solutions of the problem. It is therefore conceivable that BIEM itself provides information concerning the behavior of the solution. If it does, one may then utilize the information thus obtained to implement an accurate BIEM. The purpose of this paper is to see if such a "feed back process" is possible in BIEM for Biot's theory.

As is usually the case in applied mechanics, an investigation of the behavior of the solution is made easier when one works with quantities having a clear physical meaning such as the spatial coordinate and time. This suggests the use of a time domain approach, although some integral transformations with respect to time may lead to a practical numerical method of solution. (See Cheng and Liggett (1984) for such an attempt.) Also, we have to deviate from practical approaches such as those proposed by Banerjee and Butterfield (1981), Kuorki *et al.* (1982) and Garcia-Suarez and Alarcon (1982) which use BIEMs for the heat equation and elastostatics. This is because the solutions of Biot's equations are known to behave differently from those of heat equations or elastostatics. Hence we are led to the so-called time domain direct BIEM. The earliest attempt at this approach was made by Predeleanu (1981) who used a potential representation for the velocity of the soil skeleton. However, this is not a very convenient choice for numerical analysis because the velocity of the soil skeleton is known to behave like Dirac's delta as a function of time when the given data jump suddenly. In addition his formulation uses the pore pressure as the initial condition, which means that his method almost always requires some volume integral evaluation; as is known, the initial pressure seldom vanishes. This consideration motivated the development of another time domain direct BIEM for Biot's theory (Nishimura, 1985), which reflects both the mathematical and physical properties of Biot's theory in a more natural manner than the predecessors do. Actually, this formulation is free of the drawbacks in Predeleanu's formulation. Also noteworthy is the recent work by Cheng and Predeleanu (1987) whose BIE formulation is analogous to the one in Nishimura (1985).

The detail of Nishimura's analysis, however, has not been published so far except in an sketchy article (Nishimura, 1987). In view of this, we attempt to present here the full account of this theory, together with a recent numerical result. Specifically, we begin this paper by constructing the potential representations of the solution of Biot's equations guided by the previous analysis (Nishimura, 1985). We then proceed to the discussion of the initial fields and the initial behavior of the pore pressure on the boundary. This investigation discloses the structure of the singularity of the fluid velocity on the boundary. Namely, we shall see that the fluid velocity on the boundary shows a $t^{-1/2}$ singularity (t: time) the multiplying factor of which is computed from certain limiting values of pore pressure. All these results are given in forms valid in the fully anisotropic cases; this generality is achieved with the help of the method of the Fourier transform. We then restrict our attention to the 2D isotropic case and discuss a numerical method based on our formulation. This paper concludes with a few remarks after showing a numerical example.

2. STATEMENT OF THE PROBLEM

The well-known equations of Biot can be written in forms familiar to soil engineers as

$$\Delta^* \mathbf{u} - \nabla p = -\mathbf{f} \tag{1}$$

$$\operatorname{div} \dot{\mathbf{u}} + m\dot{p} - \mathbf{K} \cdot \nabla\nabla p = g \tag{2}$$

where $\mathbf{u}$, p, $\mathbf{f}$, g and $\mathbf{K}$, m ($\geqslant 0$) stand for the displacement, pressure, body force, $(i - f_f \mathbf{K} \cdot \nabla \mathbf{f})$ (f_f: fraction of fluid which is assumed to be a constant; i: rate of fluid injection per unit volume), permeability tensor and the compressibility of the fluid, respectively. (See (c) in Section 8 for other versions of Biot's equations.) The symbol Δ^* stands for the Navier operator defined by

$$\Delta^* \mathbf{u} := \operatorname{div} \mathbf{C}[\nabla \mathbf{u}] \quad (\mathbf{u}\text{: vector, } \mathbf{C}[\nabla \mathbf{u}]_{ij} := C_{ijkl} \partial_k u_l) \tag{3}$$

where $\mathbf{C}$ is the elasticity tensor (fourth order) which is assumed to be constant. We also assume the usual symmetry and positive definiteness for both $\mathbf{C}$ and $\mathbf{K}$, and use for the time derivative ($\dot{} = \mathrm{d}/\mathrm{d}t$, t: time; we shall also use s to indicate a value of t).

It is plain to see that the following identity holds for any sufficiently smooth vectors $\mathbf{u}$, $\mathbf{u}^*$ and scalars p, p^*:

$$\begin{aligned}
&\int_{s_1}^{s_2} \int_{\partial D} \{\dot{\mathbf{u}}^* \cdot \mathbf{s} - \dot{\bar{\mathbf{s}}}^* \cdot \mathbf{u} - p^* \mathbf{n} \cdot \mathbf{K} \nabla p + \mathbf{n} \cdot \mathbf{K} (\nabla p^*) p\} \, \mathrm{d}S \, \mathrm{d}t \\
&= \int_{s_1}^{s_2} \int_{D} \{\dot{\mathbf{u}}^* \cdot (\Delta^* \mathbf{u} - \nabla p) - (\Delta^* \dot{\mathbf{u}}^* + \nabla \dot{p}^*) \cdot \mathbf{u} \\
&\quad + p^* (\operatorname{div} \dot{\mathbf{u}} + m\dot{p} - \mathbf{K} \cdot \nabla\nabla p) - (\operatorname{div} \dot{\mathbf{u}}^* - m\dot{p}^* - \mathbf{K} \cdot \nabla\nabla p^*) p\} \, \mathrm{d}V \, \mathrm{d}t \\
&\quad - \int_{D} [p^* (\operatorname{div} \mathbf{u} + mp)]_{s_1}^{s_2} \, \mathrm{d}V
\end{aligned} \tag{4}$$

where D is an open domain in R^N ($N = 2$ or 3) with a smooth boundary ∂D, $[\cdot]_{s_1}^{s_2} = \cdot(s_2) - \cdot(s_1)$, $\mathbf{s}$ and $\bar{\mathbf{s}}^*$ are the traction and "adjoint" surface traction defined by

$$\mathbf{s} = (\mathbf{C}[\nabla\mathbf{u}] - \mathbf{1}p)\mathbf{n}, \quad \bar{\mathbf{s}}^* = (\mathbf{C}[\nabla\mathbf{u}^*] + \mathbf{1}p^*)\mathbf{n} \tag{5a, b}$$

and s_1 and s_2 are numbers, respectively.

As suggested by eqn (4) we seek a solution of the following initial boundary value problem.

Find a regular solution $(\mathbf{u}, p)$ of eqns (1) and (2) in $D \times (t > 0)$, subject to an initial condition

$$(\text{div }\mathbf{u} + mp)|_{t=0} = \theta \quad \text{in } D \tag{6}$$

and boundary conditions for $t > 0$

$$\begin{aligned} &\mathbf{u} = \mathbf{u}_0 && \text{on } \partial D_u \\ &\mathbf{s} = \mathbf{s}_0 && \text{on } \partial D_s, \ (\overline{\partial D_u \cup \partial D_s} = \partial D, \ \partial D_u \cap \partial D_s = \phi) \\ &p = p_0 && \text{on } \partial D_p \\ &r := -\mathbf{n}\cdot\mathbf{K}\nabla p = r_0 && \text{on } \partial D_r \end{aligned}$$

$$(\overline{\partial D_p \cup \partial D_r} = \partial D, \quad \partial D_p \cap \partial D_r = \phi) \tag{7a–d}$$

where θ, $\mathbf{u}_0$, $\mathbf{s}_0$, p_0 and r_0 are given functions.

By the word "regular" we mean smoothness in x in addition to the following.

(1) $\mathbf{u}$ and p in $D \times (t > 0)$ are piecewise smooth in t for a fixed x. In addition, these functions approach zero sufficiently quickly as $|\mathbf{x}| \to \infty$ if D is unbounded.

(2) For points $\mathbf{x} \in (\partial D_u \cup \partial D_s) \cap (\partial D_p \cup \partial D_r) = \partial D_R$, $\mathbf{u}$, s and p satisfy the same smoothness condition as in (1) except that "$\mathbf{x} \in D$" is replaced by "$\mathbf{x} \in \partial D_R$". As to $\partial p/\partial n$ and r, we allow them to have singularities of the following forms:

$$\sum_{i=0}^{M} C_i(\mathbf{x})H(t-s_i)(t-s_i)^{\beta_i} \quad (s_0 = 0) \tag{8}$$

where $C_i(\mathbf{x})$ are "nice" functions, $s_i (0 \leqslant i \leqslant M)$ are the values of t where the solution suffers discontinuity, $H(\cdot)$ is the step function and β_i are constants such that

$$\beta_i > -1 \tag{9}$$

respectively. Equation (9) implies that the amount of the fluid flow through ∂D in an infinitesimal time interval must be infinitesimal.

(3) $\mathbf{u}$, s, p, $\partial p/\partial n$ on ∂D may have singularities near points where the type of boundary conditions changes, but the order of the singularity is sufficiently small that the following arguments are justified.

Accordingly, we specify the data $\mathbf{u}_0$, $\mathbf{s}_0$, p_0, q_0 and θ keeping consistency with conditions (1)–(3). As to f and g, we assume piecewise continuity in t for a fixed $\mathbf{x} \in \bar{D} = D + \partial D$.

The uniqueness of the solution to this problem is easily established by using the standard argument. Indeed, the solution to our problem is totally unique except when $\partial D_u = \phi$ where the solution may include an arbitrary rigid motion, and when $m = 0$, $\partial D_p = \partial D_s = \phi$ where p is determined to within an additional constant.

The physical meaning of the statement of our problem may be obvious except for the initial condition in eqn (6). To interpret this formula, we assume that there was a certain consolidation process going on before $t = 0$. For example we have $\mathbf{u} = \mathbf{0}$, $p = 0$ for $t < 0$ if the soil was at rest before the loading begins. One easily shows that the following holds (Nishimura, 1987)

$$(\operatorname{div} \mathbf{u}+mp)|_{t=0^-} = (\operatorname{div} \mathbf{u}+mp)|_{t=0} . \tag{10}$$

Since the left-hand side of eqn (10) is identified with the left-hand side of eqn (6), we have

$$\theta = (\operatorname{div} \mathbf{u}+mp)|_{t=0} . \tag{11}$$

In particular we have $\theta = 0$ if the soil is with a quiescent past.

3. FUNDAMENTAL SOLUTIONS AND INTEGRAL REPRESENTATION OF SOLUTION

Let $\dot{\mathbf{U}}$, $\dot{\mathbf{V}}$, $\mathbf{P}$ and Q be functions which satisfy the "causality" (i.e. $\dot{\mathbf{U}}$, etc. vanish for $t < 0$) and

$$\begin{bmatrix} \Delta^* & -\nabla\partial/\partial t \\ -\nabla & -m\partial/\partial t+\mathbf{K}\cdot\nabla\nabla \end{bmatrix}\begin{bmatrix} \dot{\mathbf{U}} & \dot{\mathbf{V}} \\ \mathbf{P} & Q \end{bmatrix} = -\begin{bmatrix} \mathbf{1}\delta(t)\delta(\mathbf{x}) & 0 \\ 0 & \delta(t)\delta(\mathbf{x}) \end{bmatrix} \tag{12}$$

where $\delta(\cdot)$ is the Dirac delta. We call $\dot{\mathbf{U}}$, $\dot{\mathbf{V}}$, $\mathbf{P}$ and Q the fundamental solutions for eqns (1) and (2). These fundamental solutions can easily be obtained by using the Fourier transform. Namely, we have

$$\begin{aligned}
\dot{\mathbf{U}} &= \delta(t)\mathbf{U}_0 + \mathscr{U} \\
\dot{\mathbf{V}} &= \delta(t)\mathbf{V}_0 + \mathscr{V} \\
\mathbf{P} &= -\mathrm{i}H(t)\mathfrak{F}_\xi^{-1}\left(\frac{\Delta^{*-1}(\xi)\xi}{m+\xi\cdot\Delta^{*-1}(\xi)\xi}\,\mathrm{e}^{-G(\xi)t}\right) \\
Q &= H(t)\mathfrak{F}_\xi^{-1}\left(\frac{1}{m+\xi\cdot\Delta^{*-1}(\xi)\xi}\,\mathrm{e}^{-G(\xi)t}\right)
\end{aligned} \tag{13a–d}$$

where

$$\begin{aligned}
\mathbf{U}_0 &= \mathfrak{F}_\xi^{-1}\left(\Delta^{*-1}(\xi) - \frac{\Delta^{*-1}(\xi)\xi\otimes\Delta^{*-1}(\xi)\xi}{m+\xi\cdot\Delta^{*-1}(\xi)\xi}\right) \\
\mathbf{V}_0 &= \mathfrak{F}_\xi^{-1}\left(\frac{-\mathrm{i}\Delta^{*-1}(\xi)\xi}{m+\xi\cdot\Delta^{*-1}(\xi)\xi}\right) \\
\mathscr{U} &= \mathfrak{F}_\xi^{-1}\left(H(t)\Delta^{*-1}(\xi)\xi\otimes\Delta^{*-1}(\xi)\xi\,\frac{\xi\cdot\mathbf{K}\xi}{(m+\xi\cdot\Delta^{*-1}(\xi)\xi)^2}\,\mathrm{e}^{-G(\xi)t}\right) \\
\mathscr{V} &= \mathfrak{F}_\xi^{-1}\left(H(t)\mathrm{i}\Delta^{*-1}(\xi)\xi\,\frac{\xi\cdot\mathbf{K}\xi}{(m+\xi\cdot\Delta^{*-1}(\xi)\xi)^2}\,\mathrm{e}^{-G(\xi)t}\right).
\end{aligned} \tag{14a–d}$$

Also, $\mathfrak{F}_\xi^{-1}$ indicates the Fourier inverse transform ($\xi \to \mathbf{x}$), $\Delta^{*-1}(\xi)$ the inverse of the matrix obtained by replacing ∇ in Δ^* by ξ, and

$$G(\xi) = \frac{\xi\cdot\mathbf{K}\xi}{m+\xi\cdot\Delta^{*-1}(\xi)\xi}. \tag{15}$$

For $N = 2$ one would either have to interpret the non-integrable integral included in the Fourier inversion in eqn (14a) as the finite part (Gel'fand and Shilov, 1964) or an integral performed on a path in a complex plane which circumvents the singular point of the integrand (i.e. origin) (Mizohata, 1973). Note that $\dot{\mathbf{P}}$ and $\dot{Q}$ can be expressed as

$$\dot{\mathbf{P}}(\mathbf{x},t) = \mathbf{P}_0(\mathbf{x})\delta(t) + \dot{\mathscr{P}}(\mathbf{x},t)$$

$$\dot{Q}(\mathbf{x},t) = Q_0(\mathbf{x})\delta(t) + \dot{\mathscr{Q}}(\mathbf{x},t) \tag{16a, b}$$

where

$$\mathbf{P}_0 = \mathbf{V}_0, \quad \dot{\mathscr{P}} = \dot{\mathscr{V}}$$

$$Q_0 = \mathfrak{F}_\xi^{-1}\left(\frac{1}{m+\xi\cdot\Delta^{*-1}(\xi)\xi}\right)$$

$$\dot{\mathscr{Q}} = -H(t)\mathfrak{F}_\xi^{-1}\left(\frac{\xi\cdot\mathbf{K}\xi}{(m+\xi\cdot\Delta^{*-1}(\xi)\xi)^2}\,\mathrm{e}^{-G(\xi)t}\right). \tag{17a–d}$$

Equations (16) and (17) are obtained with the help of eqns (13c) and (13d).

The Fourier transforms of $\mathbf{U}_0$, $\mathbf{V}_0$, $\mathbf{P}_0$ and Q_0 are homogeneous functions of order -2, -1, -1 and 0, respectively. The theory of Fourier transforms of homogeneous functions then tells the following (Mizohata, 1973).

(i) $\mathbf{U}_0$ is a homogeneous function of order -1 when $N = 3$. When $N = 2$, $\mathbf{U}_0$ has an estimate

$$|\mathbf{U}_0(\mathbf{x})| \leqslant C_1 + C_2 \log|\mathbf{x}| \quad (C_1, C_2\text{: constants}) \tag{18}$$

near the origin.

(ii) $\mathbf{V}_0$, $\mathbf{P}_0$ and $\nabla\mathbf{U}_0$ are homogeneous functions of order $1-N$.

(iii) Q_0 can be expressed as

$$Q_0(\mathbf{x}) = C_Q\delta(\mathbf{x}) + \text{v.p.}\,\dot{Q}_0(\mathbf{x}) \tag{19}$$

where

$$C_Q = \frac{1}{|S_N|}\int_{S_N} \mathfrak{F}_x Q_0(\xi)\,\mathrm{d}S$$

$$\text{v.p.}\,\dot{Q}_0(\mathbf{x}) = \mathfrak{F}_\xi^{-1}\{\mathfrak{F}_x Q_0(\xi) - C_Q\} \tag{20a, b}$$

S_N is the N-dimensional unit sphere, $|S_N|$ its surface area and v.p. indicates Cauchy's principal value, respectively. Also, $\dot{Q}_0(\mathbf{x})$ is a homogeneous function of order $-N$ the mean value of which over S_N vanishes. We may sometimes write Q_0 for $\dot{Q}_0$ since they are identical except at the origin. $\nabla\mathbf{V}_0$ also has an expression analogous to eqn (19).

We next consider the time-dependent kernels ($\mathscr{U}$, etc.) in eqns (13) and (14). A useful observation is that these kernels are written as

$$\mathfrak{F}_\xi^{-1}\hat{F}(\xi)\,\mathrm{e}^{-G(\xi)t} = \frac{1}{(2\pi)^N}\int_{R^N}\hat{F}(\xi)\,\mathrm{e}^{-G(\xi)t}\,\mathrm{e}^{\mathrm{i}\xi\cdot\mathbf{x}}\,\mathrm{d}\xi, \quad t > 0 \tag{21}$$

where $\hat{F}(\xi)$ is a certain homogeneous function of order n $(-1 \leqslant n \leqslant 2)$ ($n = -1$ for $\mathbf{P}$, 2 for $\nabla\mathscr{V}$, etc.). Writing this integral as $I(\mathbf{x},t)$ we can show that $I(\mathbf{x},t)$ is a smooth function for a positive t which satisfies

$$|I(\mathbf{x},t)| \leqslant \text{const.}\,\frac{1}{t^{1/2}|\mathbf{x}|^{n+N-1}} \tag{22}$$

for $|\mathbf{x}| \neq 0$. This proves that $\mathscr{U}$, $\nabla\mathscr{U}$, $\mathscr{V}$, $\nabla\mathscr{V}$, $\mathbf{P}$, $\mathscr{P}$, $\nabla\mathbf{P}$, Q, $\mathscr{Q}$ and ∇Q are smooth functions for a non-zero t and are integrable in t for a fixed $|\mathbf{x}| \neq 0$. We also note that from eqns (13c), (13d), (17a) and (17c) it follows that

$$(\mathbf{P}(\mathbf{x}, t), Q(\mathbf{x}, t)) \to (\mathbf{P}_0(\mathbf{x}), Q_0(\mathbf{x})) \quad \text{as} \quad t \downarrow 0. \tag{23}$$

We now proceed to the construction of the potential representations for $\mathbf{u}$ and p. To this end, we introduce functions $\dot{\mathbf{U}}^*$, $\dot{\mathbf{V}}^*$, $\mathbf{P}^*$ and Q^* defined as

$$\begin{bmatrix} \dot{\mathbf{U}}^*(\mathbf{x}, t) & \dot{\mathbf{V}}^*(\mathbf{x}, t) \\ \mathbf{P}^*(\mathbf{x}, t) & Q^*(\mathbf{x}, t) \end{bmatrix} = \begin{bmatrix} -\dot{\mathbf{U}}(-\mathbf{x}, -t) & -\dot{\mathbf{V}}(-\mathbf{x}, -t) \\ \mathbf{P}(-\mathbf{x}, -t) & Q(-\mathbf{x}, -t) \end{bmatrix}. \tag{24}$$

These functions are easily seen to satisfy the "anti-causality" (i.e. $\dot{\mathbf{U}}^*$, etc. vanish for $t > 0$) and

$$\begin{bmatrix} \Delta^* & \nabla\partial/\partial t \\ \nabla & -m\partial/\partial t - \mathbf{K}\cdot\nabla\nabla \end{bmatrix} \begin{bmatrix} \dot{\mathbf{U}}^* & \dot{\mathbf{V}}^* \\ \mathbf{P}^* & Q^* \end{bmatrix} = \begin{bmatrix} \mathbf{1}\delta(t)\delta(\mathbf{x}) & 0 \\ 0 & \delta(t)\delta(\mathbf{x}) \end{bmatrix}. \tag{25}$$

We then substitute $\dot{\mathbf{U}}^*$, etc. into $\dot{\mathbf{u}}^*$, etc. in eqn (4) with the help of eqn (24). This process yields potential representations of the solution to eqns (1) and (2) in the following forms:

$$\begin{aligned} \tilde{\mathbf{u}}(\mathbf{x}, s) = & \int_{\partial D} \mathbf{U}_0(\mathbf{x}-\mathbf{y})\mathbf{s}(\mathbf{y}, s)\, \mathrm{d}S - \int_{\partial D} \mathbf{S}_0(\mathbf{x}, \mathbf{y})\mathbf{u}(\mathbf{y}, s)\, \mathrm{d}S \\ & + \int_D \mathbf{U}_0(\mathbf{x}-\mathbf{y})\mathbf{f}(\mathbf{y}, s)\, \mathrm{d}V + \int_{\partial D}\int_0^s \dot{\mathscr{U}}(\mathbf{x}-\mathbf{y}, s-t)\mathbf{s}(\mathbf{y}, t)\, \mathrm{d}t\, \mathrm{d}S \\ & - \int_{\partial D}\int_0^s \dot{\mathscr{S}}(\mathbf{x}, \mathbf{y}, s-t)\mathbf{u}(\mathbf{y}, t)\, \mathrm{d}t\, \mathrm{d}S - \int_{\partial D}\int_0^s \mathbf{P}(\mathbf{x}-\mathbf{y}, s-t) r(\mathbf{y}, t)\, \mathrm{d}t\, \mathrm{d}S \\ & + \int_{\partial D}\int_0^s \mathbf{R}(\mathbf{x}, \mathbf{y}, s-t) p(\mathbf{y}, t)\, \mathrm{d}t\, \mathrm{d}S + \int_D\int_0^s \dot{\mathscr{U}}(\mathbf{x}-\mathbf{y}, s-t)\mathbf{f}(\mathbf{y}, t)\, \mathrm{d}t\, \mathrm{d}V \\ & + \int_D\int_0^s \mathbf{P}(\mathbf{x}-\mathbf{y}, s-t) g(\mathbf{y}, t)\, \mathrm{d}t\, \mathrm{d}V + \int_D \mathbf{P}(\mathbf{x}-\mathbf{y}, s)\theta(\mathbf{y})\, \mathrm{d}V \end{aligned} \tag{26}$$

and

$$\begin{aligned} \tilde{p}(\mathbf{x}, s) = & \int_{\partial D} \mathbf{V}_0(\mathbf{x}-\mathbf{y})\cdot\mathbf{s}(\mathbf{y}, s)\, \mathrm{d}S - \int_{\partial D} \mathbf{T}_0(\mathbf{x}, \mathbf{y})\cdot\mathbf{u}(\mathbf{y}, s)\, \mathrm{d}S \\ & + \int_D \mathbf{V}_0(\mathbf{x}-\mathbf{y})\cdot\mathbf{f}(\mathbf{y}, s)\, \mathrm{d}V + \int_{\partial D}\int_0^s \dot{\mathscr{V}}(\mathbf{x}-\mathbf{y}, s-t)\cdot\mathbf{s}(\mathbf{y}, t)\, \mathrm{d}t\, \mathrm{d}S \\ & - \int_{\partial D}\int_0^s \dot{\mathscr{T}}(\mathbf{x}, \mathbf{y}, s-t)\cdot\mathbf{u}(\mathbf{y}, t)\, \mathrm{d}t\, \mathrm{d}S - \int_{\partial D}\int_0^s Q(\mathbf{x}-\mathbf{y}, s-t) r(\mathbf{y}, t)\, \mathrm{d}t\, \mathrm{d}s \\ & + \int_{\partial D}\int_0^s W(\mathbf{x}, \mathbf{y}, s-t) p(\mathbf{y}, t)\, \mathrm{d}t\, \mathrm{d}S + \int_D\int_0^s \dot{\mathscr{V}}(\mathbf{x}-\mathbf{y}, s-t)\cdot\mathbf{f}(\mathbf{y}, t)\, \mathrm{d}t\, \mathrm{d}V \\ & + \int_D\int_0^s Q(\mathbf{x}-\mathbf{y}, s-t) g(\mathbf{y}, t)\, \mathrm{d}t\, \mathrm{d}V + \int_D Q(\mathbf{x}-\mathbf{y}, s)\theta(\mathbf{y})\, \mathrm{d}V, \quad \mathbf{x} \notin \partial D \end{aligned} \tag{27}$$

where $\mathbf{S}_0$, $\dot{\mathscr{S}}$, $\mathbf{T}_0$, $\dot{\mathscr{T}}$, $\mathbf{R}$ and W are kernels defined by

$$\begin{bmatrix} S_{0il}(\mathbf{x},\mathbf{y},s) \\ T_{0l}(\mathbf{x},\mathbf{y},s) \end{bmatrix} = \frac{\partial}{\partial y_l}\begin{bmatrix} U_{0ik}(\mathbf{x}-\mathbf{y},s) \\ V_{0k}(\mathbf{x}-\mathbf{y},s) \end{bmatrix} C_{jmkl}n_m(\mathbf{y}) + \begin{bmatrix} P_{0i}(\mathbf{x}-\mathbf{y},s) \\ Q_0(\mathbf{x}-\mathbf{y},s) \end{bmatrix} n_j(\mathbf{y})$$

$$\begin{bmatrix} \mathscr{S}_{ij}(\mathbf{x},\mathbf{y},s) \\ \mathscr{T}_j(\mathbf{x},\mathbf{y},s) \end{bmatrix} = \frac{\partial}{\partial y_l}\begin{bmatrix} \dot{\mathscr{U}}_{ik}(\mathbf{x}-\mathbf{y},s) \\ \dot{\mathscr{V}}_k(\mathbf{x}-\mathbf{y},s) \end{bmatrix} C_{jmkl}n_m(\mathbf{y}) + \begin{bmatrix} \dot{\mathscr{P}}_i(\mathbf{x}-\mathbf{y},s) \\ \dot{\mathscr{Q}}(\mathbf{x}-\mathbf{y},s) \end{bmatrix} n_j(\mathbf{y})$$

$$\begin{bmatrix} R_i(\mathbf{x},\mathbf{y},s) \\ W(\mathbf{x},\mathbf{y},s) \end{bmatrix} = -\frac{\partial}{\partial y_j}\begin{bmatrix} P_i(\mathbf{x}-\mathbf{y},s) \\ Q(\mathbf{x}-\mathbf{y},s) \end{bmatrix} K_{jk}n_k(\mathbf{y}) \tag{28a–c}$$

and

$$(\tilde{\mathbf{u}},\tilde{p}) = \begin{cases} (\mathbf{u},p) & \mathbf{x}\in D \\ (0,0) & \mathbf{x}\in D^c = R^N\backslash\bar{D}. \end{cases} \tag{29}$$

Equations (26) and (27) will be used mainly as the basis for our boundary integral equation method (BIEM) to be discussed later. However, these equations are interesting in their own right. For examples one sees that eqns (26) and (27) express $\mathbf{u}$ and p in terms of space (∂D or D) integrals and space–time integrals. The former are discontinuous with respect to time when the data in eqns (7) jump as functions of time, whereas the latter are not. This indicates that the space integrals correspond to instantaneous response, and the space time integrals describe the dependence of $\mathbf{u}$ and p on the history. In this way eqns (26) and (27) decompose the solution into terms of different physical origin.

Finally, we note that eqns (26) and (27) are totally free of volume integrals when the soil is subjected to a sudden loading under vanishing body force and no fluid injection. This "boundary only" property is one of the advantages of our formulation in numerical analysis.

4. INITIAL FIELDS

We now discuss the initial behavior of the solution to our initial boundary value problem. The result of this investigation will be used later not only for computing the initial field but also for setting the initial value for boundary quantities $\mathbf{u}$, $\mathbf{s}$ and p in BIEM. Some definitions and notation are introduced first. By "initial field" we mean

$$\lim_{s\downarrow 0} p(\mathbf{x},s), \quad \mathbf{x}\in D \tag{30}$$

etc. A limit of this form for a certain fixed point in R^N is called the initial value. The initial condition is what we "prescribe", but the initial field is what we "compute". In other words, the initial field is part of the unknowns. The "boundary value" of a certain field, say $p(\mathbf{x},s)$, indicates the limit

$$\lim_{\mathbf{x}(\in D \text{ or } D^c)\to\mathbf{x}_0} p(\mathbf{x},s) \tag{31}$$

where $\mathbf{x}_0$ is a point on the boundary. Hence the boundary value of the initial field of $p(\mathbf{x},s)$ is

$$\lim_{\mathbf{x}(\in D)\to\mathbf{x}_0}\ \lim_{s\downarrow 0} p(\mathbf{x},s) \tag{32}$$

while the initial value of the boundary value of $p(\mathbf{x},s)$ is

$$\lim_{s\downarrow 0}\ \lim_{\mathbf{x}(\in D)\to\mathbf{x}_0} p(\mathbf{x},s). \tag{33}$$

In the sequel we shall write

$$p(\mathbf{x},0) := \lim_{s \to 0} p(\mathbf{x},s), \quad (\mathbf{x}\in R^N)$$

$$p(\mathbf{x}_0,s) := \lim_{\mathbf{x}(\in D)\to \mathbf{x}_0} p(\mathbf{x},s), \quad (\mathbf{x}_0\in \partial D,\ s>0). \tag{34a, b}$$

Our tools for the investigation of the initial fields are the equations

$$\tilde{\mathbf{u}}(\mathbf{x},0) = \int_{\partial D} \mathbf{U}_0(\mathbf{x}-\mathbf{y})\mathbf{s}(\mathbf{y},0)\,dS - \int_{\partial D} \mathbf{S}_0(\mathbf{x},\mathbf{y})\mathbf{u}(\mathbf{y},0)\,dS + \int_D \mathbf{U}_0(\mathbf{x}-\mathbf{y})\mathbf{f}(\mathbf{y},0)\,dV + \int_D \mathbf{P}_0(\mathbf{x}-\mathbf{y})\theta(\mathbf{y})\,dV \tag{35}$$

and

$$\tilde{p}(\mathbf{x},0) = \int_{\partial D} \mathbf{V}_0(\mathbf{x}-\mathbf{y})\cdot\mathbf{s}(\mathbf{y},0)\,dS - \int_{\partial D} \mathbf{T}_0(\mathbf{x}-\mathbf{y})\cdot\mathbf{u}(\mathbf{y},0)\,dS + \int_D \mathbf{V}_0(\mathbf{x}-\mathbf{y})\cdot\mathbf{f}(\mathbf{y},0)\,dV + C_Q\tilde{\theta}(\mathbf{x}) + \text{v.p.}\int_D \dot{Q}_0(\mathbf{x}-\mathbf{y})\theta(\mathbf{y})\,dV, \quad \mathbf{x}\notin\partial D \tag{36}$$

which we obtain by letting $s \downarrow 0$ in eqns (26) and (27) with the help of eqns (19)–(23), where v.p. indicates the integral in the sense of Cauchy's principal value.

Equations (35) and (36) are the representations of the initial fields in terms of the initial values of the boundary values of $\mathbf{u}$ and s. It is important to note that there is no *a priori* reason to assume

$$\lim_{\mathbf{x}(\in D)\to\mathbf{x}_0} \mathbf{u}(\mathbf{x},0) = \mathbf{u}(\mathbf{x}_0,0)$$

$$\lim_{\mathbf{x}(\in D)\to\mathbf{x}_0} p(\mathbf{x},0) = p(\mathbf{x}_0,0) \quad (\mathbf{x}_0\in\partial D) \tag{37a, b}$$

at this moment. Actually, we shall see that eqn (37a) is correct, but the analysis in Section 5 proves that eqn (37b) is incorrect.

We now proceed to the computation of the boundary values of the expressions in eqns (35) and (36) with the help of eqns (A1)–(A8). (See Appendix, where we have listed several useful formulae obtained by using the methods reported in Nishimura and Kobayashi, 1987a, b.) We easily obtain

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \tilde{\mathbf{u}}(\mathbf{x},0) = \mp\tfrac{1}{2}\mathbf{u}(\mathbf{x}_0,0) + \int_{\partial D} \mathbf{U}_0(\mathbf{x}_0-\mathbf{y})\mathbf{s}(\mathbf{y},0)\,dS - \text{v.p.}\int_{\partial D} \mathbf{S}_0(\mathbf{x}_0,\mathbf{y})\mathbf{u}(\mathbf{y},0)\,dS + \int_D \mathbf{U}_0(\mathbf{x}_0-\mathbf{y})\mathbf{f}(\mathbf{y},0)\,dV + \int_D \mathbf{P}_0(\mathbf{x}_0-\mathbf{y})\theta(\mathbf{y})\,dV, \quad \mathbf{x}_0\in\partial D_R \tag{38}$$

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \tilde{p}(\mathbf{x},0) = \pm\frac{1}{2}\frac{\Delta^{*-1}(\mathbf{n})\mathbf{n}}{m+\mathbf{n}\cdot\Delta^{*-1}(\mathbf{n})\mathbf{n}}\cdot\mathbf{s}(\mathbf{x}_0,0) \pm \frac{1}{2}\left(\frac{\operatorname{div}\mathbf{u}(\mathbf{x}_0,s) - \Delta^{*-1}(\mathbf{n})\mathbf{n}\cdot\mathbf{C}[\nabla\mathbf{u}(\mathbf{x}_0,s)]\mathbf{n}}{m+\mathbf{n}\cdot\Delta^{*-1}(\mathbf{n})\mathbf{n}}\right)$$

$$+\frac{1}{2}\left(C_Q \mp \frac{1}{m+\mathbf{n}\cdot\Delta^{*-1}(\mathbf{n})\mathbf{n}}\right)\theta(\mathbf{x}_0)$$

$$+\text{v.p.}\int_{\partial D}\mathbf{V}_0(\mathbf{x}_0-\mathbf{y})\cdot\mathbf{s}(\mathbf{y},0)\,\mathrm{d}S-\text{p.f.}\int_{\partial D}\mathbf{T}_0(\mathbf{x}_0,\mathbf{y})\cdot\mathbf{u}(\mathbf{y},0)\,\mathrm{d}S$$

$$+\int_D \mathbf{V}_0(\mathbf{x}_0-\mathbf{y})\cdot\mathbf{f}(\mathbf{y},0)\,\mathrm{d}V+\text{v.p.}^-\int_D Q_0(\mathbf{x}_0-\mathbf{y})\theta(\mathbf{y})\,\mathrm{d}V,\quad \mathbf{x}_0\in\partial D_R \tag{39}$$

where p.f. indicates the finite part, v.p.$^-$ the principal value integral defined in eqn (A9) and the upper (lower) sign the approach from the exterior (interior) of D, respectively. From eqn (38) it follows that

$$\lim_{\mathbf{x}(\in D)\to\mathbf{x}_0}\mathbf{u}(\mathbf{x},0)=\lim_{\mathbf{x}(\in D)\to\mathbf{x}_0}\tilde{\mathbf{u}}(\mathbf{x},0)-\lim_{\mathbf{x}(\in D^c)\to\mathbf{x}_0}\tilde{\mathbf{u}}(\mathbf{x},0)$$

$$=\mathbf{u}(\mathbf{x}_0,0),\quad \mathbf{x}_0\in\partial D_R \tag{40}$$

where we have used eqn (29). In the same manner we use eqn (39) to obtain

$$\lim_{\mathbf{x}(\in D)\to\mathbf{x}_0}p(\mathbf{x},0)=\lim_{\mathbf{x}(\in D)\to\mathbf{x}_0}\tilde{p}(\mathbf{x},0)-\lim_{\mathbf{x}(\in D^c)\to\mathbf{x}_0}\tilde{p}(\mathbf{x},0)$$

$$=-\frac{\Delta^{*-1}(\mathbf{n})\mathbf{n}}{m+\mathbf{n}\cdot\Delta^{*-1}(\mathbf{n})\mathbf{n}}\cdot\mathbf{s}(\mathbf{x}_0,0)$$

$$-\left(\frac{\operatorname{div}\mathbf{u}(\mathbf{x}_0,s)-\Delta^{*-1}(\mathbf{n})\mathbf{n}\cdot\mathbf{C}[\nabla\mathbf{u}(\mathbf{x}_0,s)]\mathbf{n}}{m+\mathbf{n}\cdot\Delta^{*-1}(\mathbf{n})\mathbf{n}}\right)$$

$$+\frac{\theta(\mathbf{x}_0)}{m+\mathbf{n}\cdot\Delta^{*-1}(\mathbf{n})\mathbf{n}}. \tag{41}$$

Also, we can write these limits in different forms. Namely, we have

$$\lim_{\mathbf{x}(\in D)\to\mathbf{x}_0}p(\mathbf{x},0)=\lim_{\mathbf{x}(\in D)\to\mathbf{x}_0}\tilde{p}(\mathbf{x},0)+\lim_{\mathbf{x}(\in D^c)\to\mathbf{x}_0}\tilde{p}(\mathbf{x},0)$$

$$=2\left(\frac{C_Q}{2}\theta(\mathbf{x}_0)+\text{v.p.}\int_{\partial D}\mathbf{V}_0(\mathbf{x}_0-\mathbf{y})\cdot\mathbf{s}(\mathbf{y},0)\,\mathrm{d}S\right.$$

$$-\text{p.f.}\int_{\partial D}\mathbf{T}_0(\mathbf{x}_0,\mathbf{y})\cdot\mathbf{u}(\mathbf{y},0)\,\mathrm{d}S+\int_D\mathbf{V}_0(\mathbf{x}_0-\mathbf{y})\cdot\mathbf{f}(\mathbf{y},0)\,\mathrm{d}V$$

$$\left.+\text{v.p.}^-\int_D Q_0(\mathbf{x}_0-\mathbf{y})\theta(\mathbf{y})\,\mathrm{d}V\right),\quad \mathbf{x}_0\in\partial D_R \tag{42}$$

and a similar result for $\mathbf{u}$, which we shall omit here.

We next consider how one computes the initial field. As can be seen easily, one just has to determine $\mathbf{u}(\mathbf{x}_0,0)$ and $\mathbf{s}(\mathbf{x}_0,0)$ on the boundary and use eqns (35) and (36) to compute the initial field. Hence we are led to the question of how to determine $\mathbf{u}(\mathbf{x}_0,0)$ and $\mathbf{s}(\mathbf{x}_0,0)$ for $\mathbf{x}_0\in\partial D$. Equations (7a) and (7b) show that half of these quantities are given as the boundary data, but the rest of them are still to be determined. We can, however, easily obtain an integral equation to determine these quantities. Namely, we use the exterior limit in eqn (38) which gives

$$0 = -\tfrac{1}{2}\mathbf{u}(\mathbf{x}_0, 0) + \int_{\partial D} \mathbf{U}_0(\mathbf{x}_0 - \mathbf{y})\mathbf{s}(\mathbf{y}, 0)\, \mathrm{d}S$$
$$-\text{v.p.} \int_{\partial D} \mathbf{S}_0(\mathbf{x}_0, \mathbf{y})\mathbf{u}(\mathbf{y}, 0)\, \mathrm{d}S + \int_D \mathbf{U}_0(\mathbf{x}_0 - \mathbf{y})\mathbf{f}(\mathbf{y}, 0)\, \mathrm{d}V$$
$$+\int_D \mathbf{P}_0(\mathbf{x}_0 - \mathbf{y})\theta(\mathbf{y})\, \mathrm{d}V, \quad \mathbf{x}_0 \in \partial D_R. \tag{43}$$

Note that it is not necessary to consider a BIE obtained from eqn (36) for determining the initial field since we already have enough equations. This is in contrast to the subsequent analysis to be discussed in Section 5 where one has to solve two integral equations simultaneously. As a cost for this saving, however, eqn (43) loses the uniqueness of the solution when $m = 0$ and D has holes, although the solution to the original boundary value problem is unique. This is because eqn (43) coincides with a BIE for elastostatics with given volumetric strain θ: the BIE for this problem is known to be nonunique (Kobayashi and Nishimura, 1982). This purely mathematical difficulty, however, can be eliminated by using remedies discussed previously (Kobayashi and Nishimura, 1982). After solving eqn (43) for $\mathbf{u}(\mathbf{x}_0, 0)$ and $\mathbf{s}(\mathbf{x}_0, 0)$ $(\mathbf{x}_0 \in \partial D)$, one may determine the boundary values of the initial fields through eqns (40) and (41).

Finally, we remark that the initial behavior of the solution in D has been investigated so far mainly from physical viewpoints in soil mechanics and in geophysics. For example Rice and Cleary (1976) introduced some material constants so as to facilitate the physical interpretation of the undrained response, or the initial behavior in our terminology, of a fluid saturated porous media. The analysis in this section shows that eqns (26) and (27) provide a mathematical alternative for investigating the behavior of the solutions, as well as a method of computing them quantitatively.

5. BOUNDARY INTEGRAL EQUATIONS AND INITIAL PRESSURE ON THE BOUNDARY

In this section we consider the limits of eqns (26) and (27) for $s > 0$ as the observation point $\mathbf{x}$ tends to a point $\mathbf{x}_0$ on the boundary. This calculation determines the boundary integral equations for consolidation problems, which will be used as a basis for the numerical BIEM to be discussed later. We then let s (time) tend to zero in the BIEs thus obtained to determine the relation between the limits in eqns (32) and (33). The result of this computation will be used later to set the initial conditions for p and r on ∂D.

It is now a simple matter to obtain the BIEs for our problem. Actually, by putting the exterior limits in eqns (26) and (27) equal to zero with the help of eqns (A1)–(A14), we obtain

$$\tfrac{1}{2}\mathbf{u}(\mathbf{x}, s) = \int_{\partial D} \mathbf{U}_0(\mathbf{x} - \mathbf{y})\mathbf{s}(\mathbf{y}, s)\, \mathrm{d}S - \text{v.p.} \int_{\partial D} \mathbf{S}_0(\mathbf{x}, \mathbf{y})\mathbf{u}(\mathbf{y}, s)\, \mathrm{d}S$$
$$+\int_D \mathbf{U}_0(\mathbf{x} - \mathbf{y})\mathbf{f}(\mathbf{y}, s)\, \mathrm{d}V + \int_{\partial D}\int_0^s \mathscr{U}(\mathbf{x} - \mathbf{y}, s - t)\mathbf{s}(\mathbf{y}, t)\, \mathrm{d}t\, \mathrm{d}S$$
$$-\text{v.p.} \int_{\partial D}\int_0^s \mathscr{S}(\mathbf{x}, \mathbf{y}, s - t)\mathbf{u}(\mathbf{y}, t)\, \mathrm{d}t\, \mathrm{d}S - \int_{\partial D}\int_0^s \mathbf{P}(\mathbf{x} - \mathbf{y}, s - t)r(\mathbf{y}, t)\, \mathrm{d}t\, \mathrm{d}S$$
$$+\int_{\partial D}\int_0^s \mathbf{R}(\mathbf{x}, \mathbf{y}, s - t)p(\mathbf{y}, t)\, \mathrm{d}t\, \mathrm{d}S + \int_D\int_0^s \mathscr{U}(\mathbf{x} - \mathbf{y}, s - t)\mathbf{f}(\mathbf{y}, t)\, \mathrm{d}t\, \mathrm{d}V$$
$$+\int_D\int_0^s \mathbf{P}(\mathbf{x} - \mathbf{y}, s - t)g(\mathbf{y}, t)\, \mathrm{d}t\, \mathrm{d}V + \int_D \mathbf{P}(\mathbf{x} - \mathbf{y}, s)\theta(\mathbf{y})\, \mathrm{d}V \tag{44}$$

and

$$\begin{aligned}\tfrac{1}{2}p(\mathbf{x},s) &= \text{v.p.}\int_{\partial D}\mathbf{V}_0(\mathbf{x}-\mathbf{y})\cdot\mathbf{s}(\mathbf{y},s)\,\mathrm{d}S-\text{p.f.}\int_{\partial D}\mathbf{T}_0(\mathbf{x},\mathbf{y})\cdot\mathbf{u}(\mathbf{y},s)\,\mathrm{d}S\\
&+\int_D\mathbf{V}_0(\mathbf{x}-\mathbf{y})\cdot\mathbf{f}(\mathbf{y},s)\,\mathrm{d}V+\text{v.p.}\int_{\partial D}\int_0^s\mathscr{V}(\mathbf{x}-\mathbf{y},s-t)\cdot\mathbf{s}(\mathbf{y},t)\,\mathrm{d}t\,\mathrm{d}S\\
&-\text{p.f.}\int_{\partial D}\int_0^s\mathscr{T}(\mathbf{x},\mathbf{y},s-t)\cdot\mathbf{u}(\mathbf{y},t)\,\mathrm{d}t\,\mathrm{d}S-\int_{\partial D}\int_0^s Q(\mathbf{x}-\mathbf{y},s-t)r(\mathbf{y},t)\,\mathrm{d}t\,\mathrm{d}S\\
&+\text{v.p.}\int_{\partial D}\int_0^s W(\mathbf{x},\mathbf{y},s-t)p(\mathbf{y},t)\,\mathrm{d}t\,\mathrm{d}S+\int_D\int_0^s\mathscr{V}(\mathbf{x}-\mathbf{y},s-t)\cdot\mathbf{f}(\mathbf{y},t)\,\mathrm{d}t\,\mathrm{d}V\\
&+\int_D\int_0^s Q(\mathbf{x}-\mathbf{y},s-t)g(\mathbf{y},t)\,\mathrm{d}t\,\mathrm{d}V+\int_D Q(\mathbf{x}-\mathbf{y},s)\theta(\mathbf{y})\,\mathrm{d}V,\quad \mathbf{x}\in\partial D.\end{aligned}\tag{45}$$

For $s>0$ we solve eqns (44) and (45) simultaneously using eqns (7) to determine boundary quantities completely.

We now let s tend to zero in these integral equations. One readily shows, with the help of eqn (A20) and the comment below eqn (A19), that this process applied to eqn (44) yields eqn (43). On the other hand a similar calculation using eqns (45), (A15)–(A19) and (A21) leads to

$$\begin{aligned}p(\mathbf{x}_0,0) &:= \lim_{s\downarrow 0} p(\mathbf{x}_0,s)\\
&= 2\Bigg(\text{v.p.}\int_{\partial D}\mathbf{V}_0(\mathbf{x}_0-\mathbf{y})\cdot\mathbf{s}(\mathbf{y},0)\,\mathrm{d}S-\text{p.f.}\int_{\partial D}\mathbf{T}_0(\mathbf{x}_0,\mathbf{y})\cdot\mathbf{u}(\mathbf{y},0)\,\mathrm{d}S\\
&\quad+\int_D\mathbf{V}_0(\mathbf{x}_0-\mathbf{y})\cdot\mathbf{f}(\mathbf{y},0)\,\mathrm{d}V+\frac{C_Q}{2}\theta(\mathbf{x}_0)+\text{v.p.}\int_D Q_0(\mathbf{x}_0-\mathbf{y})\theta(\mathbf{y})\,\mathrm{d}V\\
&\quad-\lim_{s\downarrow 0}\lim_{\mathbf{x}\to\mathbf{x}_0}\int_{\partial D}\int_0^s Q(\mathbf{x}-\mathbf{y},s-t)r(\mathbf{y},t)\,\mathrm{d}t\,\mathrm{d}S\Bigg),\quad \mathbf{x}_0\in\partial D_R.\end{aligned}\tag{46}$$

This result establishes a relation between eqns (32) and (33). Indeed, by comparing eqns (46) and (42) we obtain

$$\lim_{\mathbf{x}(\in D)\to\mathbf{x}_0} p(\mathbf{x},0) = p(\mathbf{x}_0,0)+2\lim_{s\downarrow 0}\lim_{\mathbf{x}\to\mathbf{x}_0}\int_{\partial D}\int_0^s Q(\mathbf{x}-\mathbf{y},s-t)r(\mathbf{y},t)\,\mathrm{d}t\,\mathrm{d}S,\quad \mathbf{x}_0\in\partial D_R.\tag{47}$$

Equation (47), together with eqn (A17), rules out the possibility of $-1<\beta<-1/2$ (see eqns (8) and (A18)), because p has to be finite by assumption. Also we see that eqn (47) reduces to

$$\lim_{\mathbf{x}(\in D)\to\mathbf{x}_0} p(\mathbf{x},0) = p(\mathbf{x}_0,0)-\left(\frac{\pi\mathbf{n}\cdot\mathbf{K}\mathbf{n}}{m+\mathbf{n}\cdot\Delta^{*-1}(\mathbf{n})\mathbf{n}}\right)^{1/2}\gamma(\mathbf{x}_0)\tag{48}$$

with the help of eqn (A17), where

$$\gamma(\mathbf{x}_0) = \lim_{s \downarrow 0} \frac{\partial p}{\partial n}(\mathbf{x}_0, s)\sqrt{s}. \tag{49}$$

Hence, we conclude

$$\beta = -\tfrac{1}{2} \tag{50}$$

as long as the limits in eqns (32) and (33) are both finite and different. In this case the coefficient γ of the $t^{-1/2}$ singularity in $\partial p/\partial n$ (see eqn (49)) is obtained from eqn (48). If $\beta > -1/2$ holds, or, in particular, if $\mathbf{x}_0 \in \partial D_r$ and $r_0(\mathbf{x}_0, t)$ is bounded as a function of t (see eqn (7d)), we necessarily have

$$\lim_{\mathbf{x}(\in D) \to \mathbf{x}_0} p(\mathbf{x}, 0) = p(\mathbf{x}_0, 0), \quad \mathbf{x}_0 \in \partial D_R \tag{51}$$

because $\gamma(\mathbf{x}_0) = 0$ in eqns (48) and (49).

Finally, we remark that eqn (48) implies

$$\lim_{\mathbf{x}(\in D) \to \mathbf{x}_0} p(\mathbf{x}, 0) \neq p(\mathbf{x}_0, 0) \tag{52}$$

in general, which might look queer at first. As a matter of fact, it is not. Indeed, the process of determining expression (32) discussed in Section 4 tells that expression (32) is independent of p_0 or r_0 (see eqns (7)). Actually, expression (32) is determined only by $\mathbf{u}_0$, $\mathbf{s}_0$, $\mathbf{f}$ and θ because one obtains expression (32) by solving eqn (43) followed by the use of eqn (41). On the other hand we are supposed to specify $p(\mathbf{x}_0, 0) = p_0$ on ∂D_p arbitrarily. Therefore, $p(\mathbf{x}_0, 0)$ on ∂D_p is independent of the data in eqns (6), (7a) and (7b) and, hence, independent of eqn (41). Therefore, we generally have eqn (52).

5.1. *Example*

In the case of isotropy we have

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \quad K_{ij} = k\delta_{ij} \tag{53a, b}$$

where (λ, μ) and k are Lame's constants and the permeability constant, respectively. Furthermore, we assume $m = 0$. The one-dimensional motion

$$u_1 = u_1(x_1, t), \quad u_2 = u_3 = 0, \; p = p(x_1, t) \tag{54a–c}$$

with initial and boundary conditions (see eqns (6) and (7))

$$\left.\frac{\mathrm{d}u_1}{\mathrm{d}x_1}\right|_{t=0} = 0$$
$$s_1 = p^0 \text{ (constant)}, \quad p = 0 \text{ on } x_1 = 0$$
$$u_1 = 0, \quad \frac{\partial p}{\partial n} = 0 \quad \text{on } x_1 = h \quad (h > 0: \text{constant}) \tag{55a–c}$$

produces p given by (Terzaghi, 1943)

$$p(x_1, t) = \sum_{m=0}^{\infty} \frac{2p^0}{M} \sin \frac{Mx_1}{h} \exp(-M^2 T_v) \tag{56}$$

where $M = (2m+1)\pi/2$, $T_v = c_v t/h^2$ and

$$c_v = k(\lambda+2\mu). \tag{57}$$

A direct calculation using eqn (56) shows that

$$\lim_{x_1(>0)\to 0} p(x_1, 0) = p^0 \tag{58}$$

and

$$\frac{\partial p}{\partial n}(0, t) \sim -p^0/\sqrt{(\pi c_v t)} \quad \text{as } t\downarrow 0. \tag{59}$$

Therefore, eqn (49) gives $\gamma(0) = -p^0/\sqrt{(\pi c_v)}$. On the other hand the boundary condition at $x_1 = 0$ says

$$\lim_{t\downarrow 0} p(0, t) = 0. \tag{60}$$

Hence by using

$$\left(\frac{\mathbf{n}\cdot\Delta^{*-1}(\mathbf{n})\mathbf{n}}{\mathbf{n}\cdot\mathbf{K}\mathbf{n}}\right)^{1/2} = \sqrt{(k(\lambda+2\mu))} \tag{61}$$

which follows from eqns (3) and (53), we conclude that eqn (48) is satisfied.

6. NUMERICAL PROCEDURES

In this section we shall discuss a numerical method of analysis using the formulation developed so far. We shall restrict our attention to the 2D isotropic case characterized by eqns (53a) and (53b), where the fundamental solution is available. Also, we shall neglect the compressibility of fluid ($m = 0$) following a common practice among soil engineers.

6.1. *Fundamental solutions*

We first determine the fundamental solutions. The Fourier inversion based on eqns (14) leads to (Nishimura, 1985; Cheng and Predeleanu, 1987)

$$\begin{aligned}
\dot{\mathbf{U}} &= \frac{\delta(t)}{\pi}\left(-\frac{1}{4\mu}\log\rho + \frac{\nabla\rho\otimes\nabla\rho}{4\mu}\right) + \frac{kH(t)}{\pi}\left[\mathbf{1}\left(\frac{1-e^{-\rho^2/4c_v t}}{2\rho^2}\right)\right.\\
&\quad \left. -\nabla\rho\otimes\nabla\rho\left\{\frac{1-e^{-\rho^2/4c_v t}}{\rho^2} - \frac{e^{-\rho^2/4c_v t}}{4c_v t}\right\}\right]\\
\dot{\mathbf{V}} &= \delta(t)\frac{\nabla\rho}{2\pi\rho} - \frac{H(t)\rho\, e^{-\rho^2/4c_v t}}{8\pi c_v t^2}\nabla\rho\\
\mathbf{P} &= H(t)\left(\frac{1-e^{-\rho^2/4c_v t}}{2\pi\rho}\right)\nabla\rho\\
Q &= H(t)\frac{e^{-\rho^2/4c_v t}}{4\pi k t}
\end{aligned} \tag{62a–d}$$

where c_v is a constant defined in eqn (57), $\rho = |x-y|$, $\delta(t)$ is Dirac's delta and $H(t)$ is the step function, respectively. The ∇'s in eqns (62) are with respect to $\mathbf{x}$.

6.2. Integral equations

In view of the possible initial singularity of r on ∂D it is preferable not to use integrals involving r in eqns (44) and (45) for numerical purpose. In the present analysis, we have avoided this difficulty by using an integration by parts with respect to time. Namely, we have replaced r in eqns (44) and (45) by

$$q(\mathbf{x},s) = \int_0^s r(\mathbf{x},t)\,\mathrm{d}t \tag{63}$$

with the help of identities

$$\int_{\partial D}\int_0^s \mathbf{P}(\mathbf{x}-\mathbf{y},s-t)r(\mathbf{y},t)\,\mathrm{d}t\,\mathrm{d}S$$

$$= \text{v.p.}\int_{\partial D} \mathbf{P}_0(\mathbf{x}-\mathbf{y})q(\mathbf{y},s)\,\mathrm{d}S + \text{v.p.}\int_{\partial D}\int_0^s \dot{\mathscr{P}}(\mathbf{x}-\mathbf{y},s-t)q(\mathbf{y},s)\,\mathrm{d}t\,\mathrm{d}S$$

$$\int_{\partial D}\int_0^s Q(\mathbf{x}-\mathbf{y},s-t)r(\mathbf{y},t)\,\mathrm{d}t\,\mathrm{d}S = \int_{\partial D}\int_0^s \dot{\mathscr{Q}}(\mathbf{x}-\mathbf{y},s-t)q(\mathbf{y},s)\,\mathrm{d}t\,\mathrm{d}S, \quad \mathbf{x}\in\partial D$$

(64a, b)

where

$$\dot{\mathscr{P}}(\mathbf{x},s) = -\frac{H(s)\rho\,\mathrm{e}^{-\rho^2/4c_p s}}{8\pi c_p s^2}\nabla\rho \tag{65}$$

$$\dot{\mathscr{Q}}(\mathbf{x},s) = \frac{H(s)}{4\pi k}\left\{\frac{\rho^2\,\mathrm{e}^{-\rho^2/4c_p s}}{4c_p s^3} - \frac{\mathrm{e}^{-\rho^2/4c_p s}}{s^2}\right\}. \tag{66}$$

The apparent nonsymmetry of eqns (64a) and (64b) is due to the assumption of isotropy, with which Q_0 is seen to vanish.† (See eqn (17c).) Also the integrals in eqns (44) and (45) involving $\mathbf{S}_0$ and W are seen to converge in an ordinary sense due to the assumed isotropy and incompressibility of the fluid.

One may doubt the effectiveness of eqn (45) for numerical applications because of the apparently very strong singularity of the kernels. As a matter of fact, this is not to be the case. Indeed, the singularities in the integrands cancel with each other reducing themselves to integrable singularities, as we shall see shortly.

6.3. Numerical methods of integration

We shall now implement a numerical method of solution of Biot's equations using the formulation investigated so far.

Our method consists of the following steps (Nishimura *et al.*, 1986).

(1) We first compute the initial field. A direct BIEM for incompressible elasticity is used to this end (Kobayashi and Nishimura, 1982). Namely, we discretize eqn (43) by using boundary conditions (7a) and (7b) and certain shape functions, solve the resulting algebraic equations for discretized $\mathbf{u}(\mathbf{x}_0, 0)$ and $\mathbf{s}(\mathbf{x}_0, 0)$, and then use eqns (35) and (36) to obtain the initial fields.

We next compute the initial values of the boundary quantities $\mathbf{u}$, $\mathbf{s}$, p and q as a preparation for the subsequent analysis in step (2). The first two, however, have been obtained as solutions of eqn (43), and the initial value of q is zero by definition. As to p on ∂D_p boundary condition (7c) provides the answer. Finally, one uses eqns (48) and (7d) to

† $Q_0 = C\delta(\mathbf{x})$ to be precise, where C is a constant.

set the initial value of p on ∂D_R after computing expression (32) with the help of the solution of eqns (43) and (41). In the present context eqn (41) simplifies to

$$\lim_{y(\in D)\to x(\in \partial D)} p(\mathbf{y},0) = 2\mu\left(\theta(\mathbf{x})-\mathbf{c}\cdot\frac{\partial \mathbf{u}}{\partial c}(\mathbf{x},0)\right)-\mathbf{s}\cdot\mathbf{n} \tag{67}$$

where $\mathbf{c}$ is a unit tangent vector to ∂D at $\mathbf{x}$. As can be inferred from eqn (67) it is convenient to use a sufficiently smooth shape function for $\mathbf{u}$ since eqn (67) includes tangential differentiation.

As has been pointed out in Section 5, the integral equation for the initial field loses the uniqueness of solution when D has holes. Correspondingly, the associated numerical equation becomes ill-conditioned. We therefore avoid this inaccuracy by using one of the remedies discussed previously (Kobayashi and Nishimura, 1982).

(2) Let the solution be known up to a certain time $s-\Delta t \geqslant 0$ where $\Delta t > 0$ is a constant. We now wish to obtain the solution at $t = s(>0)$. To this end, we introduce time interpolation functions $\Omega_1(t)$ and $\Omega_2(t)$ in the interval $[s-\Delta t, s]$ in a way that

$$\Omega_1(s) = 1,\quad \Omega_1(s-\Delta t) = 0$$
$$\Omega_2(s) = 0,\quad \Omega_2(s-\Delta t) = 1 \tag{68a–d}$$

are satisfied. One may then interpolate a function of t, say $\mathbf{u}(\cdot,t)$, by

$$\mathbf{u}(\cdot,t) \sim \mathbf{u}(\cdot,s)\Omega_1(t)+\mathbf{u}(\cdot,s-\Delta t)\Omega_2(t) \tag{69}$$

in $[s-\Delta t, s]$, where $\mathbf{u}(\cdot,s-\Delta t)$ is known, but $\mathbf{u}(\cdot,s)$ may not be. It is then clear that the time integration, together with approximations of the forms in eqn (69) applied to boundary quantities such as $\mathbf{u}$, reduces eqns (44) and (45) to integral equations of the following form:

$$\int_{\partial D} (\text{kernels})\cdot(\text{quantities at } t = s)\,\mathrm{d}S = \text{known functions}. \tag{70}$$

We shall henceforth call the transformation of this type "time discretization". After time discretizing eqns (44) and (45), we apply the conventional BIE techniques using spatial shape functions. The solution to the resulting simultaneous algebraic equations determines the boundary quantities at $t = s$.

(3) Repeat step (2) by setting $s+\Delta t$ for the new s.

We next discuss our specific choice of the time interpolation function and the method of integration. We here choose linear time variation so as to keep the method simple. This time variation also allows us to calculate all the pertinent time integrals analytically. Our next question is what the singularities of the kernels in eqn (70) are. We here consider only the kernels which operate on $\mathbf{u}(\cdot,s)$ because they include the strongest singularities one sees in eqns (44) and (45). The kernel $\mathbf{K}_1(\mathbf{x},\mathbf{y})$ which operates on $\mathbf{u}(\cdot,s)$ in the time-discretized version of eqn (44) is

$$\mathbf{K}_1(\mathbf{x},\mathbf{y}) := \mathbf{S}_0(\mathbf{x},\mathbf{y})+\int_{s-\Delta t}^{s} \dot{\mathscr{S}}(\mathbf{x},\mathbf{y},s-t)\left(1-\frac{s-t}{\Delta t}\right)\mathrm{d}t \tag{71}$$

where we have used a linear time interpolation. A direct calculation using eqns (62) and (28) then shows (Nishimura, 1987)

$$\mathbf{K}_1(\mathbf{x},\mathbf{y}) = \frac{1}{2\pi(\lambda+2\mu)\rho}\{(\mu\mathbf{1}+2(\lambda+\mu)\nabla\rho\otimes\nabla\rho)(\nabla\rho\cdot\mathbf{n})+\mu(\mathbf{n}\otimes\nabla\rho-\nabla\rho\otimes\mathbf{n})\}$$
$$+O(\rho\log\rho)\quad \text{as } \rho\downarrow 0. \tag{72}$$

The singular term in this expression is seen to coincide with the elastostatic double layer kernel (Kupradze, 1965). In the same manner we obtain a similar kernel (denoted by $\mathbf{K}_2$) in the time-discretized version of eqn (45) as

$$\mathbf{K}_2(\mathbf{x},\mathbf{y}) := \mathbf{T}_0(\mathbf{x},\mathbf{y}) + \int_{s-\Delta t}^{s} \dot{\mathcal{T}}(\mathbf{x},\mathbf{y},s-t)\left(1-\frac{s-t}{\Delta t}\right) \mathrm{d}t$$

$$= -\frac{\mu}{2\pi c_i \Delta t}\mathbf{n}\log\rho + O(1) \tag{73}$$

for a small ρ. This shows that $\mathbf{K}_2(\mathbf{x},\mathbf{y})$ has an integrable singularity at $\rho = 0$. As a matter of fact, the method of Fourier transform used previously (Nishimura and Kobayashi, 1987a, b) shows that these conclusions concerning the singularities of $\mathbf{K}_1$ and $\mathbf{K}_2$ remain valid in the general N-dimensional anisotropic case. We shall, however, not go into further detail on this topic.

Now that we have shown the singularities of our kernels to be the same as those of elasticity, we may apply the techniques for evaluating the elastostatic singular integrals to our integral equations. Namely, we may compute singular integrals in eqns (44) and (45) so that the discretized versions of eqns (44) and (45) are satisfied by some known solutions of eqns (1) and (2). In elastostatics rigid translations have turned out to be a convenient choice for the "known solutions" (Lachat and Watson, 1976). In consolidation, we may use:

Rigid translation

$$\mathbf{u} = \mathbf{u}_R \cdot f(t), \quad p = 0 \tag{74a, b}$$

where $\mathbf{u}_R$ is a constant vector, and $f(t)$ an arbitrary function of time.

Uniform shear

$$\mathbf{u} = (\text{uniform simple shear}) \cdot f(t), \quad p = 0. \tag{75a, b}$$

This satisfies eqns (1) and (2) because div $\mathbf{u} = 0$.

Uniform pressure

$$\mathbf{u} = \mathbf{0}, \quad p = \text{constant}. \tag{76a, b}$$

p may be an arbitrary function of time if m vanishes.

Radial expansion

$$\mathbf{u} = \frac{\mathbf{x}}{|\mathbf{x}|^2} f(t), \quad p = 0. \tag{77a, b}$$

This is a solution of eqns (1) and (2) when the material is isotropic.

Expressions (74)–(76) are interior solutions of eqns (1) and (2), while eqns (77) give an exterior solution of eqns (1) and (2) if the origin is out of the domain D. In this statement "exterior solution" means a solution which is regular in an exterior domain, and "interior solution" stands for a solution regular in a bounded domain. In using the method of substitution one has to remember that an interior (exterior) solution satisfies integral equations for interior (exterior) problems. One may, however, use an exterior solution for computing singular integrals in interior problems and vice versa. Indeed, one first adds $\mathbf{u}(p)$ to the right-hand side of eqn (44) (eqn (45)) to obtain the BIE for an exterior problem defined in the complement of D when D is bounded. The obtained BIE is satisfied by an exterior solution. The method of substitution then computes some singular integrals in the

BIE for exterior problems. However, this process computes singular integrals in the BIE for interior problems at the same time because the exterior BIE and the interior BIE differ only by non-integral terms. Note that we now have more known solutions than the number of singular integrals (two in 2D) in eqns (44) and (45). As it turned out, however, use of this technique to some of the logarithmically singular integrals increases the accuracy of the numerical solution considerably. Hence we shall try to use as many of the above four solutions as possible in the numerical example to follow.

7. NUMERICAL EXAMPLE

In this section we shall test the performance of the present method by solving a sample problem. We consider a circular hole (radius = a) in an infinite plane. The infinite plane is assumed to have a uniform initial stress $\boldsymbol{\tau}^0$ and a vanishing initial pressure. We then "excavate" a hole in a way that the following conditions are satisfied:

Initial condition

$$\theta = 0. \tag{78}$$

Boundary conditions

$$s = 0, \quad p = 0 \quad \text{on } \partial D. \tag{79a, b}$$

A standard analysis shows that the displacement on the boundary is written as

$$u_1 = \frac{a \cos \Theta}{4\mu}(\tau^0_{11} + \tau^0_{22} + (\tau^0_{11} - \tau^0_{22})F(t))$$

$$u_2 = \frac{a \sin \Theta}{4\mu}(\tau^0_{11} + \tau^0_{22} - (\tau^0_{11} - \tau^0_{22})F(t)) \tag{80}$$

where

$$F(t) = -\frac{1}{2\pi i}\int_{\varepsilon - i\infty}^{\varepsilon + i\infty} \frac{e^{\omega t}}{\omega}\left(\frac{(\lambda + 3\mu)H^{(1)}_2(\kappa a) + \mu H^{(1)}_0(\kappa a)}{(\lambda + \mu)H^{(1)}_2(\kappa a) - \mu H^{(1)}_0(\kappa a)}\right) d\omega$$

$$\kappa = \sqrt{(i\omega/c_v)} \tag{81a, b}$$

ε is a small positive number, and $H^{(1)}_i(\cdot)$ is the Hankel function of the first kind. The Cartesian axes used here are directed in the principal directions of $\boldsymbol{\tau}^0$, and the angle Θ is measured from the x_1-axis. In the present analysis we have set $\tau^0_{11} = -0.4p^0$, $\tau^0_{22} = -0.8p^0$, $\tau^0_{12} = 0$, and ν (Poisson's ratio) = 0 where p^0 is a constant. We have used 32 linear iso-parametric boundary time elements of equal length. As we have found, this problem is particularly sensitive to the accuracy of numerical integrations included in the BIEM algorithm. Therefore, we have used the method of substitution with eqns (74), (75) and (77) in order to compute not only v.p. integrals, but also some of the logarithmically singular integrals.

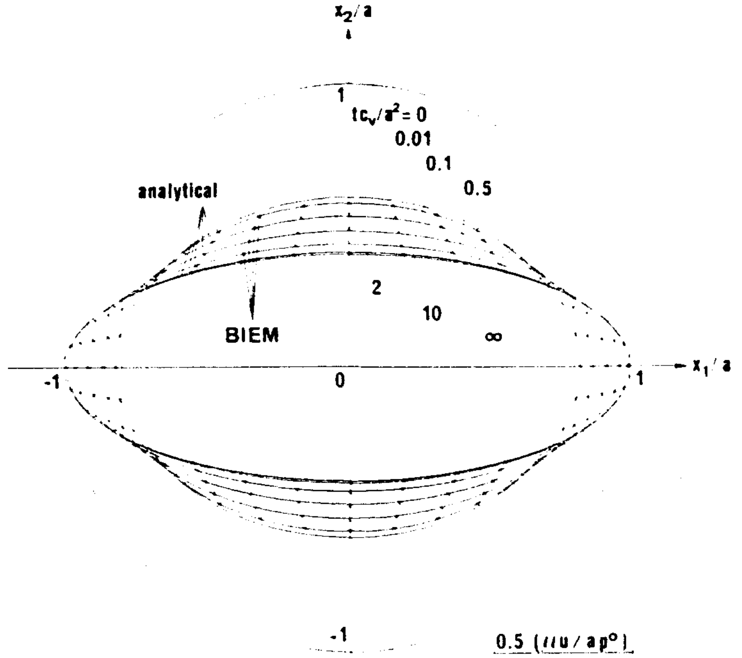

Fig. 1. Deformation of a circular hole in a prestressed infinite plane. Poisson's ratio = 0.

Figure 1 shows the deformation of the boundary for several values of t. The out-most circle represents the undeformed shape of the boundary. The lines in this figure show the results obtained from eqn (80). The Fourier inversion involved in eqns (81) has been carried out with the help of FFT with 2048 data. The symbols in the same figure indicate two series of BIEM results, i.e. the displacements on ∂D for $0 < c_v t/a^2 < 1$ obtained with $c_v \Delta t/a^2 = 1/100$ and those for $1 < c_v t/a^2 < 10$ obtained with $c_v \Delta t/a^2 = 1/10$. This figure shows that the accuracy of our BIEM is satisfactory. The CPU time for an analysis in 100 time steps was about 17 s with the VP400 of the Data Processing Center, Kyoto University. Our computer code uses the fact that the pressure on the boundary vanishes, but is otherwise completely general without taking advantage of the symmetry of the problem.

8. CONCLUDING REMARKS

(a) The main results of this paper are summarized as follows.

(1) The solution to Biot's equations has the potential representation given by eqns (26) and (27).
(2) The initial field, given by eqns (35) and (36), is computed with the help of the BIE in eqn (43).
(3) $\partial p/\partial n$ on ∂D may have a singularity proportional to $t^{-1/2}$ as shown in eqn (48). Equation (48) is useful also for determining the initial value of p on ∂D.
(4) The present formulation provides an accurate BIEM.

(b) It is easy to extend the numerical method in Section 6 to the 3D case. The fundamental solutions for the 3D case are given in Cleary (1977) (see also Rudnicki (1981)), Cheng and Predeleanu (1987) and Nishimura (1987).

(c) There are several versions of linearized Biot's equations. For example, Biot's theory in its original form yields

$$\Delta^* \mathbf{u} - \alpha \nabla p = \mathbf{0}$$

$$\beta \,\mathrm{div}\, \dot{\mathbf{u}} + m\dot{p} - \mathbf{K} \cdot \nabla\nabla p = 0 \qquad (82a, b)$$

in our notation (Biot, 1941), where α and β are constants. In eqns (82), we have omitted the inhomogeneous terms (body force and fluid injection) for the sake of simplicity. The formulation in Rice and Cleary (1976) also leads to eqns (82). Our governing equations, eqns (1) and (2), with an assumption of $m = 0$ are used by many researchers and practitioners. For example McNamee and Gibson (1960) use eqns (1) and (2) in slightly different forms. However, the difference between eqns (82) and eqns (1) and (2) is not very important mathematically because eqns (82) are equivalent to eqns (1) and (2). Indeed, the following replacement reduces eqns (82) to eqns (1) and (2) (see eqn (7)):

$$\frac{\mathbf{C}}{\alpha} \to \mathbf{C}, \quad \frac{m}{\beta} \to m, \quad \frac{\mathbf{K}}{\beta} \to \mathbf{K}, \quad \frac{\mathbf{s}_0}{\alpha} \to \mathbf{s}_0, \quad \frac{r_0}{\beta} \to r_0. \tag{83a–e}$$

This shows that our analysis applies to various versions of Biot's theory with a minor modification.

(d) Biot's theory of consolidation is closely related to coupled thermoelasticity in that these theories are based on similar equations. Sladek and Sladek (1983) investigated several BIEMs in various theories of linear thermoelasticity.

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APPENDIX. FORMULAE FOR VARIOUS LIMITS OF POTENTIALS IN CONSOLIDATION THEORY

This appendix summarizes several formulae for various limits of the potentials in Biot's theory. Interested readers are referred to Nishimura and Kobayashi (1987a, b) for the proofs.

(a) *Space integrals with time-independent kernels*

The results in Nishimura and Kobayashi (1987a) yield the following:

$$\lim_{\mathbf{x}\to\mathbf{x}_0}\int_{\partial D}\mathbf{U}_0(\mathbf{x}-\mathbf{y})\mathbf{s}(\mathbf{y},s)\,\mathrm{d}S=\int_{\partial D}\mathbf{U}_0(\mathbf{x}_0-\mathbf{y})\mathbf{s}(\mathbf{y},s)\,\mathrm{d}S \tag{A1}$$

$$\lim_{\mathbf{x}\to\mathbf{x}_0}\int_{\partial D}\mathbf{S}_0(\mathbf{x},\mathbf{y})\mathbf{u}(\mathbf{y},s)\,\mathrm{d}S=\pm\tfrac{1}{2}\mathbf{u}(\mathbf{x}_0,s)+\text{v.p.}\int_{\partial D}\mathbf{S}_0(\mathbf{x}_0-\mathbf{y})\mathbf{u}(\mathbf{y},s)\,\mathrm{d}S \tag{A2}$$

$$\lim_{\mathbf{x}\to\mathbf{x}_0}\int_{D}\mathbf{U}_0(\mathbf{x}-\mathbf{y})\mathbf{f}(\mathbf{y},s)\,\mathrm{d}V=\int_{D}\mathbf{U}_0(\mathbf{x}_0-\mathbf{y})\mathbf{f}(\mathbf{y},s)\,\mathrm{d}V \tag{A3}$$

$$\lim_{\mathbf{x}\to\mathbf{x}_0}\int_{D}\mathbf{P}_0(\mathbf{x}-\mathbf{y})\theta(\mathbf{y})\,\mathrm{d}V=\int_{D}\mathbf{P}_0(\mathbf{x}_0-\mathbf{y})\theta(\mathbf{y})\,\mathrm{d}V \tag{A4}$$

$$\lim_{\mathbf{x}\to\mathbf{x}_0}\int_{\partial D}\mathbf{V}_0(\mathbf{x}-\mathbf{y})\cdot\mathbf{s}(\mathbf{y},s)\,\mathrm{d}S=\pm\frac{1}{2}\frac{\Delta^{*-1}(\mathbf{n})\mathbf{n}}{m+\mathbf{n}\cdot\Delta^{*-1}(\mathbf{n})\mathbf{n}}\cdot\mathbf{s}(\mathbf{x}_0,s)+\text{v.p.}\int_{\partial D}\mathbf{V}_0(\mathbf{x}_0-\mathbf{y})\cdot\mathbf{s}(\mathbf{y},s)\,\mathrm{d}S \tag{A5}$$

$$\lim_{\mathbf{x}\to\mathbf{x}_0}\int_{\partial D}\mathbf{T}_0(\mathbf{x},\mathbf{y})\cdot\mathbf{u}(\mathbf{y},s)\,\mathrm{d}S=\mp\frac{1}{2}\left(\frac{\operatorname{div}\mathbf{u}(\mathbf{x}_0,s)-\Delta^{*-1}(\mathbf{n})\mathbf{n}\cdot\mathbf{C}[\nabla\mathbf{u}(\mathbf{x}_0,s)]\mathbf{n}}{m+\mathbf{n}\cdot\Delta^{*-1}(\mathbf{n})\mathbf{n}}\right)+\text{p.f.}\int_{\partial D}\mathbf{T}_0(\mathbf{x},\mathbf{y})\cdot\mathbf{u}(\mathbf{y},s)\,\mathrm{d}S \tag{A6}$$

$$\lim_{\mathbf{x}\to\mathbf{x}_0}\int_{D}\mathbf{V}_0(\mathbf{x}-\mathbf{y})\cdot\mathbf{f}(\mathbf{y},s)\,\mathrm{d}V=\int_{D}\mathbf{V}_0(\mathbf{x}_0-\mathbf{y})\cdot\mathbf{f}(\mathbf{y},s)\,\mathrm{d}V \tag{A7}$$

$$\lim_{\mathbf{x}\to\mathbf{x}_0}\int_{D}Q_0(\mathbf{x}-\mathbf{y})\theta(\mathbf{y})\,\mathrm{d}V=\frac{1}{2}\left(C_Q\mp\frac{1}{m+\mathbf{n}\cdot\Delta^{*-1}(\mathbf{n})\mathbf{n}}\right)\theta(\mathbf{x}_0)+\text{v.p.}\int_{D}Q_0(\mathbf{x}_0-\mathbf{y})\theta(\mathbf{y})\,\mathrm{d}V,\quad \mathbf{x}_0\in\partial D \tag{A8}$$

where the upper (lower) sign indicates the approach from the exterior (interior) of D, v.p. the principal value integral defined by

$$\text{v.p.}\int_{D}\cdot\,\mathrm{d}V:=\lim_{\varepsilon\downarrow 0}\int_{D\setminus B_\varepsilon(\mathbf{x}_0)}\cdot\,\mathrm{d}V \tag{A9}$$

$B_\varepsilon(\mathbf{x}_0)$ a ball having a radius ε and centered at $\mathbf{x}_0$, and C_Q a number defined in eqn (20a), respectively. Note that the apparent dependence of the non-integral term in eqn (A6) on $\partial\mathbf{u}/\partial n$ is resolved by substituting $\nabla\mathbf{u}-\mathbf{n}\otimes(\partial\mathbf{u}/\partial n)$ into $\nabla\mathbf{u}$ in eqn (A6). Also, the limiting values of these integrals as $s\downarrow 0$ are obtained by putting $s=0$ on the right-hand sides of eqns (A1)–(A8).

(b) *Space–time integrals*

By using the method in Nishimura and Kobayashi (1987a), we can prove the following results for $s>0$:

$$\lim_{\mathbf{x}\to\mathbf{x}_0}\int_{\partial D}\int_0^s\mathscr{U}(\mathbf{x}-\mathbf{y},s-t)\mathbf{s}(\mathbf{y},t)\,\mathrm{d}t\,\mathrm{d}S=\int_{\partial D}\int_0^s\mathscr{U}(\mathbf{x}_0-\mathbf{y},s-t)\mathbf{s}(\mathbf{y},t)\,\mathrm{d}t\,\mathrm{d}S \tag{A10}$$

$$\lim_{\mathbf{x}\to\mathbf{x}_0}\int_{\partial D}\int_0^s\mathscr{S}(\mathbf{x},\mathbf{y},s-t)\mathbf{u}(\mathbf{y},t)\,\mathrm{d}t\,\mathrm{d}S=\text{v.p.}\int_{\partial D}\int_0^s\mathscr{S}(\mathbf{x}_0,\mathbf{y},s-t)\mathbf{u}(\mathbf{y},t)\,\mathrm{d}t\,\mathrm{d}S \tag{A11}$$

$$\lim_{\eta\downarrow 0}\lim_{\mathbf{x}\to\mathbf{x}_0}\int_{\partial D}\int_0^s\mathscr{V}(\mathbf{x}-\mathbf{y},s-t)\cdot\mathbf{s}(\mathbf{y},t)\,\mathrm{d}t\,\mathrm{d}S=\mp\frac{1}{2}\left(\frac{\Delta^{*-1}(\mathbf{n})\mathbf{n}}{m+\mathbf{n}\cdot\Delta^{*-1}(\mathbf{n})\mathbf{n}}\right)\cdot\mathbf{s}(\mathbf{x}_0,s)$$
$$+\text{v.p.}\int_{\partial D}\int_0^s\mathscr{V}(\mathbf{x}_0-\mathbf{y},s-t)\cdot\mathbf{s}(\mathbf{y},t)\,\mathrm{d}t\,\mathrm{d}S \tag{A12}$$

$$\lim_{\mathbf{x}\to\mathbf{x}_0}\int_{\partial D}\int_0^s\mathscr{T}(\mathbf{x},\mathbf{y},s-t)\cdot\mathbf{u}(\mathbf{y},t)\,\mathrm{d}t\,\mathrm{d}S=\pm\frac{1}{2}\left(\frac{\operatorname{div}\mathbf{u}(\mathbf{x}_0,s)-\Delta^{*-1}(\mathbf{n})\mathbf{n}\cdot\mathbf{C}[\nabla\mathbf{u}(\mathbf{x}_0,s)]\mathbf{n}}{m+\mathbf{n}\cdot\Delta^{*-1}(\mathbf{n})\mathbf{n}}\right)$$
$$+\text{p.f.}\int_{\partial D}\int_0^s\mathscr{T}(\mathbf{x}_0,\mathbf{y},s-t)\cdot\mathbf{u}(\mathbf{y},t)\,\mathrm{d}t\,\mathrm{d}S \tag{A13}$$

$$\lim_{\mathbf{x}\to\mathbf{x}_0}\int_{\partial D}\int_0^s W(\mathbf{x},\mathbf{y},s-t)p(\mathbf{y},t)\,\mathrm{d}t\,\mathrm{d}S = \mp\tfrac{1}{2}p(\mathbf{x}_0,s)+\mathrm{v.p.}\int_{\partial D}\int_0^s W(\mathbf{x}_0,\mathbf{y},s-t)p(\mathbf{y},t)\,\mathrm{d}t\,\mathrm{d}S,\quad \mathbf{x}_0\in\partial D. \tag{A14}$$

Other potentials in eqns (26) and (27) satisfy relations similar to eqn (A10).

We next compute the limits of these limits as s tends to zero. One has to be careful in doing this because of the possible singularity of r as $s\downarrow 0$. A computation taking this singularity into consideration yields (Nishimura and Kobayashi, 1987b)

$$\lim_{s\downarrow 0}\lim_{\mathbf{x}\to\mathbf{x}_0}\int_{\partial D}\int_0^s \mathcal{V}(\mathbf{x}-\mathbf{y},s-t)\cdot\mathbf{s}(\mathbf{y},t)\,\mathrm{d}t\,\mathrm{d}S = \mp\frac{1}{2}\left(\frac{\Delta^{*-1}(\mathbf{n})\mathbf{n}}{m+\mathbf{n}\cdot\Delta^{*-1}(\mathbf{n})\mathbf{n}}\right)\cdot\mathbf{s}(\mathbf{x}_0,0) \tag{A15}$$

$$\lim_{s\downarrow 0}\lim_{\mathbf{x}\to\mathbf{x}_0}\int_{\partial D}\int_0^s \mathcal{T}(\mathbf{x},\mathbf{y},s-t)\cdot\mathbf{u}(\mathbf{y},t)\,\mathrm{d}t\,\mathrm{d}S = \pm\frac{1}{2}\left(\frac{\mathrm{div}\;\mathbf{u}(\mathbf{x}_0,s)-\Delta^{*-1}(\mathbf{n})\mathbf{n}\cdot\mathbf{C}[\nabla\mathbf{u}(\mathbf{x}_0,s)]\mathbf{n}}{m+\mathbf{n}\cdot\Delta^{*-1}(\mathbf{n})\mathbf{n}}\right) \tag{A16}$$

$$\lim_{s\downarrow 0}\lim_{\mathbf{x}\to\mathbf{x}_0}\int_{\partial D}\int_0^s Q(\mathbf{x}-\mathbf{y},s-t)r(\mathbf{y},t)\,\mathrm{d}t\,\mathrm{d}S = \begin{cases}\text{divergent}, & -1<\beta<-1/2\\ -(\pi\mathbf{n}\cdot\mathbf{K}\mathbf{n})^{1/2}\gamma(\mathbf{x}_0)/2(m+\mathbf{n}\cdot\Delta^{*-1}(\mathbf{n})\mathbf{n})^{1/2}, & \beta=1/2\\ 0, & -1/2<\beta\end{cases} \tag{A17}$$

where β and $\gamma(\mathbf{x}_0)$ are the exponent of the lowest power and the corresponding coefficient of the asymptotic expansion of $(\partial p/\partial n)(\mathbf{x}_0,t)$ near $t=0$, i.e.

$$\frac{\partial p}{\partial n}(\mathbf{x}_0,t) = t^{\beta}\gamma(\mathbf{x}_0)+o(t^{\beta})\quad\text{as } t\downarrow 0 \tag{A18}$$

$$\lim_{s\downarrow 0}\lim_{\mathbf{x}\to\mathbf{x}_0}\int_{\partial D}\int_0^s W(\mathbf{x},\mathbf{y},s-t)p(\mathbf{y},t)\,\mathrm{d}t\,\mathrm{d}S = \mp\tfrac{1}{2}p(\mathbf{x}_0,0),\quad \mathbf{x}_0\in\partial D. \tag{A19}$$

The limits of this form for other potentials in eqns (26) and (27) are seen to vanish.

(c) *Volume integrals with time-dependent kernels*

Finally we have (Nishimura and Kobayashi, 1987b)

$$\lim_{s\downarrow 0}\lim_{\mathbf{x}\to\mathbf{x}_0}\int_D \mathbf{P}(\mathbf{x}-\mathbf{y},s)\theta(\mathbf{y})\,\mathrm{d}V = \int_D \mathbf{P}_0(\mathbf{x}_0-\mathbf{y})\theta(\mathbf{y})\,\mathrm{d}V \tag{A20}$$

$$\lim_{s\downarrow 0}\lim_{\mathbf{x}\to\mathbf{x}_0}\int_D Q(\mathbf{x}-\mathbf{y},s)\theta(\mathbf{y})\,\mathrm{d}V = \frac{C_Q}{2}\theta(\mathbf{x}_0)+\mathrm{v.p.}\int_D Q_0(\mathbf{x}_0-\mathbf{y})\theta(\mathbf{y})\,\mathrm{d}V,\quad \mathbf{x}_0\in\partial D \tag{A21}$$

where C_Q and v.p. are defined in eqns (20a) and (A9), respectively.